REGULARIZATION SCHEMES FOR APPROXIMATE SOLUTION OF NONLINEAR REVERSE PROBLEM OF HEAT CONDUCTION

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A numerical procedure is presented for determining nonstationary boundary conditions; it is based on interval-by-interval regularization of linearized reverse problem of heat conduction.

To determine the boundary conditions for nonstationary heat exchange one has in many problems to solve the reverse problem for the heat-conduction equation with time-dependent coefficients. In a number of cases the assumption that the coefficients are constant may produce considerable errors in the determination of heat fluxes or temperatures on the surface of a solid body. In [1] direct numerical methods were proposed for solving nonlinear reverse problems of heat conduction. Stability of the solution procedure was achieved by introducing constraints on the time-step (the "step" regularization of solution). In this article a numerical algorithm is considered for restoring the boundary conditions in an one dimensional reverse problem with physical heat characteristics which are discontinuous in the coordinate and are temperature dependent; a similar condition is not employed in the algorithm.

Our approach is based essentially on linearization over time intervals of the original formulation with subsequent regularization of the linear problem using the scheme proposed in [2, 3].

Let us consider the quasilinear equation of heat conduction in a solid body,

$$\rho C \frac{\partial T}{\partial \tau} = \frac{\partial}{\partial x}\left(\lambda \frac{\partial T}{\partial x}\right), \quad \lambda = \lambda(T), \quad \rho = \rho(T), \quad C = C(T). \tag{1}$$

It is assumed that a second boundary-value problem is formulated and that an algorithm is available for its solution. To be specific let us consider the often encountered case of a two-layer lamina whose one side is subjected to a variable heat flux $q(\tau)$ and the other is thermally insulated (a transition to nonstationary second boundary condition causes no difficulties). The initial temperature profile is known.

Following [4] we write down the finite-difference analog of the problem which corresponds to the approximation of the heat-conduction equation by an implicit 6-point scheme:

$$Au = B, \tag{2}$$

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$$A = \begin{bmatrix}
d^{(1)}_{0,n} & 2e^{(1)}_{0,n} & & & & & & & & \\
e^{(1)}_{1,n} & d^{(1)}_{1,n} & e^{(1)}_{1,n} & & & & & & & \\
 & e^{(1)}_{2,n} & d^{(1)}_{2,n} & e^{(1)}_{2,n} & & & & 0 & & \\
 & & \cdot & \cdot & & & & & & \\
 & & & & & & & & & \\
 & & e^{(1)}_{n_1-1,n} & d^{(1)}_{n_1-1,n} & e^{(1)}_{n_1-1,n} & & & & & \\
 & & & h_{1,n} & d_{n_1,n} & h_{2,n} & & & & \\
 & & & & e^{(2)}_{n_1+1,n} & d^{(2)}_{n_1+1,n} & e^{(2)}_{n_1+1,n} & & & \\
 & & & & & e^{(2)}_{n_1+2,n} & d^{(2)}_{n_1+2,n} & e^{(2)}_{n_1+2,n} & & \\
 & & & & & & \cdot & \cdot & & \\
 & & 0 & & & & & e^{(2)}_{\varkappa,n} & d^{(2)}_{\varkappa,n} & e^{(2)}_{\varkappa,n} \\
 & & & & & & & & \cdot & \cdot \\
 & & & & & & & & & 2e^{(2)}_{k,n} \; d^{(2)}_{k,n}
\end{bmatrix},$$

$$U = \begin{bmatrix} T_{0,n+1} \\ T_{1,n+1} \\ T_{2,n+1} \\ \vdots \\ T_{n_1-1,n+1} \\ T_{n_1,n+1} \\ T_{n_1+1,n+1} \\ T_{n_1+2,n+1} \\ \vdots \\ T_{\varkappa,n+1} \\ \vdots \\ T_{k,n+1} \end{bmatrix}, \qquad B = \begin{bmatrix} F_{0,n} \\ b^{(1)}_{1,n} \\ b^{(1)}_{2,n} \\ \vdots \\ b^{(1)}_{n_1-1,n} \\ T_{n_1,n} \\ b^{(2)}_{n_1+1,n} \\ b^{(2)}_{n_1+2,n} \\ \vdots \\ b^{(2)}_{\varkappa,n} \\ \vdots \\ T_{k,n} \end{bmatrix},$$

where

$$d^{(\gamma)}_{p,n} = 1 + 2\,\frac{a^{(\gamma)}_{p,n}\Delta\tau}{(\Delta_\gamma x)^2}, \quad e^{(\gamma)}_{p,n} = -\,\frac{a^{(\gamma)}_{p,n}\Delta\tau}{(\Delta_\gamma x)^2},$$

$$h_{\gamma,n} = -\,\frac{2\Delta\tau}{\Delta_\gamma x}\,\frac{\lambda^{(\gamma)}_{n_1,n}}{C^{(1)}_{n_1,n}\rho^{(1)}_{n_1,n}\Delta_1 x + C^{(2)}_{n_1,n}\rho^{(2)}_{n_1,n}\Delta_2 x},$$

$$d_{n_1,n} = 1 + 2\Delta\tau\,\frac{\lambda^{(1)}_{n_1,n}/\Delta_1 x + \lambda^{(2)}_{n_1,n}/\Delta_2 x}{C^{(1)}_{n_1,n}\rho^{(1)}_{n_1,n}\Delta_1 x + C^{(2)}_{n_1,n}\rho^{(2)}_{n_1,n}\Delta_2 x},$$

$$F_{0,n} = T_{0,n} + \left(\frac{2a^{(1)}_{0,n}\Delta\tau}{\lambda^{(1)}_{0,n}\Delta_1 x}\right) q_{n+1},$$

$$b^{(\gamma)}_{s,n} = T_{s,n} + \frac{a^{(\gamma)}_{s,n}\Delta\tau}{(\Delta_\gamma x)^2}\left(\frac{\dot\lambda^{(\gamma)}_{s,n}}{4\lambda^{(\gamma)}_{s,n}}\right)(T_{s+1,n} - T_{s-1,n})^2,$$

$$T_{i,n} = T(x_i,\ \tau_n),$$

$$\lambda^{(\gamma)}_{i,n} = \lambda^{(\gamma)}(T_{i,n}),\ \rho^{(\gamma)}_{i,n} = \rho^{(\gamma)}(T_{i,n});\ C^{(\gamma)}_{i,n} = C^{(\gamma)}(T_{i,n});\ a^{(\gamma)}_{i,n} = a^{(\gamma)}(T_{i,n}),$$

$$q_n = q(\tau_n), \quad \dot\lambda^{(\gamma)}_{i,n} \equiv \frac{d\lambda^{(\gamma)}_{i,n}}{dT},$$

$$\Delta\tau = \frac{\tau_m}{m}, \quad \Delta_\gamma x = \frac{b_\gamma}{n_\gamma},$$

$$n = 1, 2, \ldots, m, \; i = 0, 1, \ldots, n_1, \ldots, k, \; n_1 + n_2 = k,$$
$$p = 0, 1, \ldots, k, \; p \neq n_1,$$
$$s = 1, 2, \ldots, k-1, \; s \neq n_1,$$
$$\gamma = 1; 2, \; T_{i,0} = \varphi(x_i).$$

Thus, if the heat flux $q(\tau)$ is known one can find the temperature field $T(x, \tau)$ of the lamina provided the following data are given: τ_m, m, b_1, n_1, b_2, n_2, $\varphi(x)$, $\lambda^{(1)}(T)$, $\rho^{(1)}(T)$, $C^{(1)}(T)$, $\lambda^{(2)}(T)$, $\rho^{(2)}(T)$, $C^{(2)}(T)$. If it is necessary to reach the required accuracy of $T(x, \tau)$ an iterative process in the coefficients is organized at each step.

Let us now consider one of the feasible ways of using the regularization method [5] to solve a reverse problem when one seeks the unknown heat flow $q(\tau)$ from the given temperature $T_\varkappa(\tau)$ at a point $i = \varkappa$.

On the time interval $0-\tau_m$ under consideration the interval $\Delta\tau_N = \tau_{n=N-1} - \tau_{n=N+l-1}$ of length $l\Delta\tau$ ($1 \leq N \leq m-l+1$, $l > 0$ is an integer) is considered and on it one linearizes the temperature determination by assuming that the physical heat coefficients are functions of the coordinate and correspond to the temperature profile at the instant $\tau = \tau_{N-1}$:

$$\lambda_{i,n}^{(\gamma)} = \lambda^{(\gamma)}(T_{i,N-1}), \; \rho_{i,n}^{(\gamma)} = \rho^{(\gamma)}(T_{i,N-1}), \; C_{i,n}^{(\gamma)} = C^{(\gamma)}(T_{i,N-1}); \; \gamma = 1, 2.$$

Then in agreement with the general superposition principle the temperature $T_\varkappa$ is represented as a sum of solutions for zero initial conditions and for zero initial temperature distributions respectively,

$$T_\varkappa(\tau) = T_\varkappa(\tau)\Big|_{\substack{q=0 \\ \psi=T_{N-1}(x)}} + T_\varkappa(\tau)\Big|_{\substack{q=q(\tau) \\ \psi=0}}. \tag{3}$$

By transition from a single thermal effect on the finite time interval to a difference form of the expression (3) [6, 7] the following system of linear algebraic equations with triangular matrix ($j = 1$ corresponds to $n = N$) is obtained for the determination of the grid function q_n on a linearization portion:

$$\sum_{j=1}^{J} \varphi_j^N q_j = f_J^N, \; J = 1, 2, \ldots, l, \tag{4}$$

where

$$\varphi_l^N = T_{\varkappa j}\Big|_{\substack{q_{j=1}=1, q_{j>1}=0 \\ \psi_i = T_{i,N-1}}} - T_{\varkappa j}\Big|_{\substack{q_{j-1}=0 \\ \psi_i = T_{i,N-1}}},$$

$$f_J^N = T_{\varkappa J} - T_{\varkappa J}\Big|_{\substack{q_{j\geq 1}=0 \\ \psi_i = T_{i,N-1}}},$$

$\tilde{T}_\varkappa$ denotes the "experimental" temperature.

One adopts the following for the regularization functional:

$$\Phi[q_j, \alpha] = \sum_{J=1}^{l}\Big(\sum_{j=1}^{J} \varphi_j^N q_j - f_J^N\Big)^2 + \alpha \sum_{j=1}^{l} (q_{j+1} - q_j)^2.$$

The partial derivatives of Φ with respect to q_j are now obtained and are made equal to zero. This results in l equations for $l + 2$ unknowns. To solve the problem the boundary conditions must be added.

It is assumed that for the first series of points $n = 1, \ldots, l$ ($N = 1$) one has

$$q_{j=0} = q_{j=1}, \tag{5}$$
$$q_{j=l} = q_{j=l+1}. \tag{6}$$

For the subsequent series of points ($N > 1$) one sets

$$q_{j=0} = q_{n=N-1}, \tag{7}$$
$$q_{j=l} = q_{j=l+1}. \tag{8}$$

The relations (5), (6) and (8) represent "natural boundary conditions." The value $q_{n=N-1}$ is the extreme left value of the heat flux which is known from the preceding regularizing system.

For a chosen interval $\Delta\tau_N$ this results in a system of algebraic equations similar to the system (16) of [2]; if it is solved for a given value of the regularization parameter α one obtains the function $q^\alpha(\tau)$. To form the coefficients and the right-hand sides of the system the original system (2) for $q_n = 0$ and $q_{n=N} = 1$, $q_{n>N} = 0$ respectively ($n = N, \ldots, N + l-1$) must be solved $2l$ times.

The choice of the regularization parameter α depends on the specified error level [8, 9, 10],

$$\varkappa(\alpha) = \delta,$$

where δ is the error of the input data which in turn depends on the time interval under consideration,

$$\varkappa(\alpha) = \begin{cases} \rho_\alpha = \sum_{J=1}^{J} \left(\sum_{j=1}^{J} \varphi_j^N q_j - f_J^N \right)^2, \\ \varphi_\alpha = \rho_\alpha + \alpha \sum_{j=1}^{l} (q_{j+1} - q_j)^2. \end{cases}$$

Since the original problem of finding the boundary conditions has been reduced by our algorithm of an iterative solution of a number of separate linear problems and since, moreover, in each such problem one has to determine the optimal value of α therefore it is important that the computer time required to automate the determination of the parameter be shortened in accordance with the error principle in the form of an equality.

It is not advisable to choose α by the inner convergence of the regularized approximations since the automation of such a process would be difficult.

One selects from the obtained q_J^α the first M values ($I < M < l$) which one adopts for the sought $q^\alpha_{J \le M} = q_n (N \le n \le N + M-1)$. The remaining values are rejected and will be determined in subsequent calculations.

Having computed the heat fluxes q_n the direct problem of finding the temperature field $T_{i,n} (N \le n \le N + M-1)$ is solved in accordance with the algorithm (2) described above. The temperature profile $T_{i, N+M-1}$ is adopted as the initial distribution to solve a similar regularizing system on the subsequent time portion.

It is advisable that for the first points in time the order of the regularizing system l be higher than for the subsequent groups of points. Since the conditions at the ends of the interval are not known one has to adopt the "natural" boundary conditions $q' = 0$ which results in the solution "deviating" from the true one in the neighborhood of these ends. The approximation obtained in the middle of the interval is sufficiently exact. Therefore, the first values q_J up to the midinterval can be regarded as the sought values and need not be improved in further calculations (all the more since in the initial stage the properties hardly change).

The choice of the lengths of intervals which can be unequal and of the quantities $\Delta\tau_N$ should depend on the problem under consideration and on the expected solution. They can be larger for body characteristics which change slowly with temperature and with a slight change of the external heat load; one should naturally choose smaller ones for intensive changes of heat fluxes towards the body with the properties depending strongly on temperature (this is especially valid for M).

We shall now discuss briefly another approach to a numerical solution of the reverse problem under consideration. For $\tau = 0$ the left ends of the time intervals are kept fixed, and the right ends, as previously, will change by the quantity $\Delta\tau_N$. The representation (3) remains valid in this case provided one takes into account that physical heat properties are regarded as functions of the coordinate and of time with the exception of the last interval $\Delta\tau_N$ in which they are calculated from the temperature profile at $\tau = \tau_{N-1}$.

It is advisable to construct the regularizing numerical algorithm under consideration by using a "trial" solution q^*; for the latter one should, of course, adopt the function $q^\alpha(\tau)$ obtained by solving the regularization system on the preceding time interval; on the last step $\Delta\tau_N$ one can set $q^*(\tau) = q^\alpha(\tau_{N-1}) = \text{const}$. Thus discrete values of the sought function are calculated in our scheme iteratively with continuous improvements until instant τ_m, and it is not necessary to join together the solutions when passing from one time interval to another. However, with the parameter N increasing the order of the regularization system grows, and as a result the total cost of computer time is higher than in the previous scheme.

Since the proposed schemes for solving nonlinear reverse problems are based on interval—after-interval regularization of the linearized problem they can be transferred to a generalized heat-conduction equation, $\rho C(\partial T/\partial \tau) = \partial/\partial x[\lambda(\partial T/\partial x)] + K(\partial T/\partial x) + Q$. In this case the finite-difference equivalent of the equation requires modification and one assumes that on the regularization interval the coefficient K is a function only of x (by analogy with λ, ρ and C). The source of heat Q on the extension of this time interval is approximated by a linear function of temperature.

Our procedure can be extended to solve the reverse problem with a moving boundary without involving any major modifications provided its motion is known.

NOTATION

a	is the temperature-conductivity coefficient;
b_1, b_2	are the thickness of 1st and 2nd lamina layers;
C	is the specific heat;
m	is the number of time steps;
n_1, n_2	are the number of space steps for 1st and 2nd layers of the lamina;
q	is the heat flux to the body due to heat conduction;
q^α	is the regularized solution of the reverse (inverse) heat-conduction problem;
T	is the temperature;
x	is the space coordinate;
α	is the regularization parameter;
γ	is the number of lamina layers;
$\Delta_1 x$, $\Delta_2 x$	is the space steps for 1st and 2nd lamina layers;
$\Delta\tau$	is the time step;
$\Delta\tau_N$	is the regularization interval;
λ	is the heat-conductivity coefficient;
ρ	is the density;
τ	is the time;
τ_m	is the right-end value of the entire time interval;
$\varphi(x)$	is the initial temperature distribution in body.

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